Indian Statistical Institute, Bangalore.
Mid-semester Exam : Measure-theoretic Probability

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Max. points : 30. **Time Limit : 3 hours.**

Answer any three questions only.

Give complete proofs. Please cite/quote appropriate results from class or assignments properly. You are also allowed to use results from other problems in the question paper.

1. Let μ be a non-negative and finitely additive set function on a semiring $\mathcal{A}$ with $\mu(\emptyset) = 0$. Let $A, A_1, \ldots, A_n$ be sets in $\mathcal{A}$. Show the following.
   (a) If A_i are disjoint and $\cup_{i=1}^n A_i \subset A$, then $\sum_{i=1}^n \mu(A_i) \leq \mu(A)$. (**4**)
   (b) If $A \subset \cup_{i=1}^n A_i$, then $\mu(A) \leq \sum_{i=1}^n \mu(A_i)$. (**4**)
   (c) Without using the semi-ring extension theorem, show that μ is countably additive iff μ is countably subadditive. (**2**)

2. (a) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show that $\mathcal{A} = \{A : \mu(A) < \infty\}$ is a semi-ring and closed under finite unions. Is $\mathcal{A}$ an algebra ? (**5**).
   (b) Let $(\Omega_j, \mathcal{F}_j)_{j=1,\ldots,n}$ be measurable spaces. Fix $(x_1, \ldots, x_n) \in \prod_{i=1}^n \Omega_i$ and $B \in \bigotimes_{i=1}^n \mathcal{F}_i$. Show that for all $1 \leq j \leq n$, the sets $\{(y_1, \ldots, y_j) \in \prod_{i=1}^j \Omega_i : (y_1, \ldots, , y_j, x_{j+1}, \ldots, x_n) \in B\}$ are $\bigotimes_{i=1}^j \mathcal{F}_i$-measurable. (**5**)

3. Let $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ be a measure space and set $a_n := \mu(\{n\})$. Let $f : \mathbb{N} \to \mathbb{R}$. Show that $\sum_n f(n)a_n = \int f \mathrm{d}\mu$ whenever either one of them is well-defined. (**10**)

4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_n : \Omega \to [0, \infty], n \geq 1$ be a sequence of simple functions such that $f_n \uparrow f$ pointwise. Show that $\lim_{n\to\infty} \int f_n \mathrm{d}\mu = \int f \mathrm{d}\mu$. (**10**).

5. (a) Let $(\Omega, \mathcal{F})$ be a measurable space with $\mathcal{F} = \sigma(\mathcal{A})$ where $\mathcal{A}$ is a semi-ring and further let Ω be a countable disjoint union of $\mathcal{A}$ sets. Let μ, ν be measures on $(\Omega, \mathcal{F})$ such that for all $A \in \mathcal{A}$, we have that $\mu(A) \leq \nu(A) < \infty$. Then show that $\mu(B) \leq \nu(B)$ for all $B \in \mathcal{F}$. (**5**)
   (b) Let μ be a set function on the semi-ring $\{(a, b] : -\infty < a \leq b < \infty\}$ such that $\mu(a, b] = b - a$. Show that the outer measure $\mu^* : 2^{\mathbb{R}} \to [0, \infty]$ is translation invariant i.e., for any $A \subset \mathbb{R}$ and $x \in \mathbb{R}$, $\mu^*(A + x) = \mu^*(A)$. (**5**)